Disentangling mass and angle dependance in neutrino mixing

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Outline

- Flavor mixing in Quantum Field Theory (QFT)
- Decomposition of the mixing generator[1]
- Conclusions

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Neutrino mixing in Quantum Field Theory (QFT)

Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\,\nu_1(x) + \cos\theta\,\nu_2(x)$$

ν_1, ν_2 are fields with definite masses.

Mixing transformations connect the two quadratic forms:

$$\mathcal{L} = \bar{\nu}_1\,(i\,\partial\!\!\!/ - m_1)\,\nu_1 + \bar{\nu}_2\,(i\,\partial\!\!\!/ - m_2)\,\nu_2$$

and

$$\mathcal{L} = \bar{\nu}_e\,(i\,\partial\!\!\!/ - m_e)\,\nu_e + \bar{\nu}_\mu\,(i\,\partial\!\!\!/ - m_\mu)\,\nu_\mu - m_{e\mu}\,(\bar{\nu}_e\,\nu_\mu + \bar{\nu}_\mu\,\nu_e)$$

with $m_e = m_1\cos^2\theta + m_2\sin^2\theta$, $m_\mu = m_1\sin^2\theta + m_2\cos^2\theta$, $m_{e\mu} = (m_2 - m_1)\sin\theta\cos\theta$.

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Mixing relations can be written as[2]

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\, \nu_1^\alpha(x)\, G_\theta(t)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\, \nu_2^\alpha(x)\, G_\theta(t)$$

with the mixing generator[3]

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

[2] M.Blasone and G.Vitiello, Annals Phys. (1995)

[3] For ν_e, we get $\frac{d^2}{d\theta^2}\, \nu_e^\alpha = -\nu_e^\alpha$ with initial conditions $\nu_e^\alpha|_{\theta=0} = \nu_1^\alpha \quad , \frac{d}{d\theta}\nu_e^\alpha\Big|_{\theta=0} = \nu_2^\alpha$.

Where ν_i are free Dirac field operators:

$$\nu_i(x) = \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \left[\, u^r_{\mathbf{k},i}(t)\, \alpha^r_{\mathbf{k},i} \; + \; v^r_{-\mathbf{k},i}(t)\, \beta^{r\dagger}_{-\mathbf{k},i} \,\right], \quad i = 1, 2.$$

Anticommutation, orthonormality and completeness relations are the standard ones

The vacuum $|0\rangle_{1,2}$ is not invariant under the action of the generator $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\, |0\rangle_{1,2}$$

Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V\to\infty} {}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty} e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \ln\left(1-\sin^2\theta\,|V_{\mathbf{k}}|^2\right)^2} = 0$$

with

$$|V_{\mathbf{k}}|^2 \equiv \sum_{r,s} |\, v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\,|^2 \neq 0 \quad for \quad m_1 \neq m_2$$

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Condensation density:

$$_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; _{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; \sin^2\theta\;|V_{\mathbf{k}}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

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Mixing transformation = Rotation nested with a Bogoliubov transformation[4].

[4] $U_{\mathbf{k}}(t) = u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\; e^{i(\omega_{k,2}-\omega_{k,1})t}$; $|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$

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Neutrino oscillation formula (exact result)[5]:

$$
\begin{aligned}
\mathcal{Q}_{\mathbf{k},\nu_e}(t) &= 1 - |U_{\mathbf{k}}|^2 \sin^2(2\theta)\,\sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\,t\right) \\
&\quad - |V_{\mathbf{k}}|^2 \sin^2(2\theta)\,\sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\,t\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_{\mathbf{k},\nu_\mu}(t) &= |U_{\mathbf{k}}|^2 \sin^2(2\theta)\,\sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\,t\right) \\
&\quad + |V_{\mathbf{k}}|^2 \sin^2(2\theta)\,\sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\,t\right)
\end{aligned}
$$

For $k \gg \sqrt{m_1 m_2}$, $|U_{\mathbf{k}}|^2 \to 1$ and $|V_{\mathbf{k}}|^2 \to 0$.

[5] M.Blasone, P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

Decomposition of mixing generator

Mixing generator function of m_1, m_2, and θ.

$$\begin{aligned} G_\theta(t) &= \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right] \\ &= \exp\Big\{ \sum_r \Big(U^*_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},2} - \epsilon^r\ V^*_{\mathbf{k}}\ \beta^r_{-\mathbf{k},1}\alpha^r_{\mathbf{k},2} + \epsilon^r\ V_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + U_{\mathbf{k}}\ \beta^r_{-\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\Big) \\ &- \sum_r \Big(U_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1} + \epsilon^r\ V^*_{\mathbf{k}}\ \beta^r_{-\mathbf{k},2}\alpha^r_{\mathbf{k},1} - \epsilon^r\ V_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1} + U^*_{\mathbf{k}}\ \beta^r_{-\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big)\Big\} \end{aligned}$$

Our aim is to disentangle the mass dependence from the one by the mixing angle.

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G_\theta(t) &= \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right] \\
&= \exp\Big\{ \sum_r \Big(U_\mathbf{k}^* \, \alpha_{\mathbf{k},1}^{r\dagger}\alpha_{\mathbf{k},2}^{r} - \epsilon^r \, V_\mathbf{k}^* \, \beta_{-\mathbf{k},1}^{r}\alpha_{\mathbf{k},2}^{r} + \epsilon^r \, V_\mathbf{k} \, \alpha_{\mathbf{k},1}^{r\dagger}\beta_{-\mathbf{k},2}^{r\dagger} + U_\mathbf{k} \, \beta_{-\mathbf{k},1}^{r}\beta_{-\mathbf{k},2}^{r\dagger}\Big) \\
&- \sum_r \Big(U_\mathbf{k} \, \alpha_{\mathbf{k},2}^{r\dagger}\alpha_{\mathbf{k},1}^{r} + \epsilon^r \, V_\mathbf{k}^* \, \beta_{-\mathbf{k},2}^{r}\alpha_{\mathbf{k},1}^{r} - \epsilon^r \, V_\mathbf{k} \, \alpha_{\mathbf{k},2}^{r\dagger}\beta_{-\mathbf{k},1}^{r\dagger} + U_\mathbf{k}^* \, \beta_{-\mathbf{k},2}^{r}\beta_{-\mathbf{k},1}^{r\dagger}\Big) \Big\}
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&- \sum_r \Big(U_\mathbf{k} \, \alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1} + \epsilon^r \, V_\mathbf{k}^* \, \beta^r_{-\mathbf{k},2}\alpha^r_{\mathbf{k},1} - \epsilon^r \, V_\mathbf{k} \, \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1} + U_\mathbf{k}^* \, \beta^r_{-\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big)\Big\}
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Let us define[6]:

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r}\left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^{r}_{-\mathbf{k},2}\right)e^{i\psi_k} + -\left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^{r}_{-\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\},$$

$$B_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r}\Theta_{\mathbf{k},i}\ \epsilon^r\left[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\}, \quad i = 1,2$$

Since $[B_1, B_2] = 0$ we put $B(\Theta_1,\Theta_2) \equiv B_1(\Theta_1)\, B_2(\Theta_2)$

6

$$\psi_k = (\omega_{k,1} - \omega_{k,2})\, t); \qquad \phi_{k,i} = 2\,\omega_{k,i}\, t; \qquad \Theta_{\mathbf{k},i} = \frac{1}{2}\cot^{-1}\left(\frac{|\mathbf{k}|}{m_i}\right)$$

The $B_i(\Theta_{\mathbf{k},i})$, $i = 1, 2$ are ordinary Bogoliubov transformations which introduce a mass shift, and $R(\theta)$ is a rotation.

Their action on the vacuum is given by:

$$|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} = \prod_{\mathbf{k},r} \left[\cos\Theta_{\mathbf{k},i} + \epsilon^r \sin\Theta_{\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i}\right]|0\rangle_{1,2}$$

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R^{-1}(\theta)|0\rangle_{1,2} &= |0\rangle_{1,2} \, .
\end{aligned}
$$

We find:

$$G_\theta \;=\; B(\Theta_1,\Theta_2)\;\; R(\theta)\;\; B^{-1}(\Theta_1,\Theta_2)$$

which is realized when the $\Theta_{\mathbf{k},i}$ are chosen as:

$$U_{\mathbf{k}} \;=\; e^{-i\psi_k}\;\cos(\Theta_{\mathbf{k},1}-\Theta_{\mathbf{k},2})$$
$$V_{\mathbf{k}} \;=\; e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}}\;\sin(\Theta_{\mathbf{k},1}-\Theta_{\mathbf{k},2})$$

Then we rewrite the generator as

$$G(\theta, m_1, m_2) = B^{-1}(m_1, m_2)\, R(\theta)\, B(m_1, m_2)$$

Moreover,

$$|0\rangle_{e,\mu} \equiv G^{-1}|0\rangle_{1,2} = |0\rangle_{1,2} + \left[B(m_1, m_2)\, ,\, R^{-1}(\theta)\right] |\widetilde{0}\rangle_{1,2}$$

which shows that the condensate structure of the flavor vacuum arises as a consequence of the non vanishing commutator $[B, R^{-1}]$.

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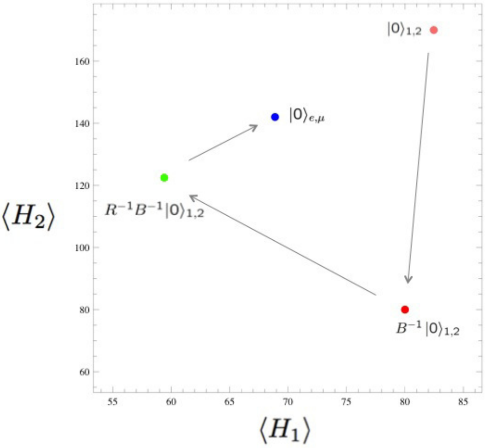
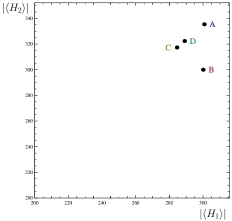
In the following table we report the vacuum expectation values of the (unordered) Hamiltonian on the various *vacua* obtained by acting step-by-step with the above generators.

$\langle H_{\mathbf{k},1} + H_{\mathbf{k},2} \rangle$	State
$-(\omega_{k,1} + \omega_{k,2})$	$\vert 0\rangle_1 \otimes \vert 0\rangle_2 \equiv \vert 0\rangle_{1,2}$
$-(k + k)$	$B^{-1}(m_1, m_2)\vert 0\rangle_{1,2} \equiv \vert\widetilde{0}\rangle_{1,2}$
$-k(2\cos^2\theta + (\frac{\omega_{k,1}}{\omega_{k,2}} + \frac{\omega_{k,2}}{\omega_{k,1}})\sin^2\theta)$	$R^{-1}(\theta)B^{-1}(m_1, m_2)\vert 0\rangle_{1,2} = R^{-1}(\theta)\vert\widetilde{0}\rangle_{1,2}$
$-(\omega_{k,1} + \omega_{k,2})\big(1 - 2\sin^2\theta\sin^2(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\big)$	$B(m_1, m_2)R^{-1}(\theta)B^{-1}(m_1, m_2)\vert 0\rangle_{1,2} \equiv \vert 0\rangle_{e,\mu}$

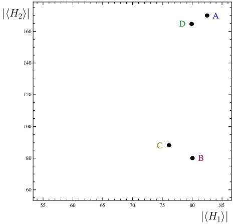


Plot of vacuum energies for H_1 and H_2 for the different vacua given in Table for

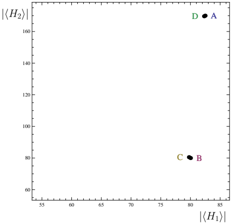
$\theta = \pi/6,\ m_1 = 20,\ m_2 = 150,\ k = 80.$



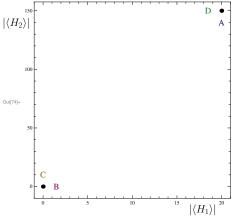
Plot of vevs of H_1 and H_2 for $\theta = \pi/4$, $k = 300$, $m_1 = 20$, $m_2 = 150$.



Plot of vevs of H_1 and H_2 for $\theta = \pi/10$, $k = 80$, $m_1 = 20$, $m_2 = 150$.



Plot of vevs of H_1 and H_2 for $\theta = \pi/40$, $k = 80$, $m_1 = 20$, $m_2 = 150$.



Plot of vevs of H_1 and H_2 for $\theta = \pi/4$, $k = 0$, $m_1 = 20$, $m_2 = 20$.

Conclusion

- Mixing transformations are not trivial in Q.F.T. $\Leftrightarrow$ they are associated to inequivalent representations.
- The mixing Generator can be seen as a rotation transformed under a Bogoliubov transformation.
- The seed of the inequivalence between the two vacua (mass and flavor) is in the non commutativity between the rotation and the Bogoliubov transformation

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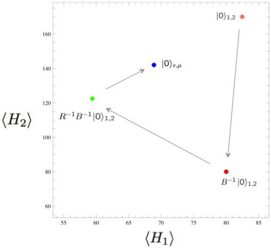

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work in progress : Can we consider a "thermodynamical" interpretation of the energies associated to the different vacua (mass and flavour)?

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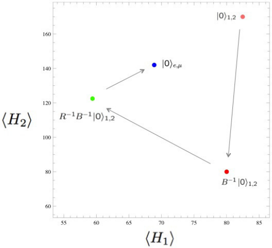

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Thank you
for your kind attention

Work in progress

According to Thermal Filed Theory (TFT) a state such as $|0(\vartheta)\rangle_i = B(\vartheta)|0\rangle$ can be written as

$$|0(\vartheta)\rangle = exp\left(-\frac{S_\alpha}{2}\right)|I\rangle = exp\left(-\frac{S_\beta}{2}\right)|I\rangle$$

with

$$|I\rangle \equiv \exp\left(\sum_{\mathbf{k},r}(-1)^{i+1}\epsilon^r e^{i\phi}\beta^{r\dagger}_{-\mathbf{k}}\alpha^{r\dagger}_{\mathbf{k}}\right)|0\rangle$$

$$S_\alpha = -\sum_{\mathbf{k},r}\left(\alpha^{r\dagger}_{\mathbf{k}}\alpha^r_{\mathbf{k}}\log\sin^2\vartheta_{\mathbf{k}} + \alpha^r_{\mathbf{k}}\alpha^{r\dagger}_{\mathbf{k}}\log\cos^2\vartheta_{\mathbf{k}}\right)$$

A similar expression holds for S_β. It is known[7] that S_α (or S_β) can be interpreted as the entropy function associated to the vacuum condensate.
With

$$n_k = \langle 0(\vartheta)|\alpha^{r\dagger}_{\mathbf{k}}\alpha^r_{\mathbf{k}}|0(\vartheta)\rangle = \sin^2(\vartheta_{\mathbf{k}})$$

we have that the expectation value of the entropy operator is

$$S_{\mathbf{k}} = \langle 0(\vartheta)|S_{\alpha_{\mathbf{k}}}|0(\vartheta)\rangle = n_{\mathbf{k}}log(n_{\mathbf{k}}) + (1-n_{\mathbf{k}})log(1-n_{\mathbf{k}})$$

[7] Y. Takahashi, H. Umezawa, Int. Jour. Mod. Phy. B (1996) 1755-1805

At this point it is natural to write a formally analogue expression for S_{ik} , entropy associated to the state we obtain after the first Bogoliubov transformation and $S_{\sigma k}$, entropy associated to the last Bogoliubov transformation.

$$S_{\sigma k} = n_{\sigma k} log(n_{\sigma k}) + (1 - n_{\sigma k}) log(1 - n_{\sigma k})$$

with

$$n_{\sigma k} = \sin^2\theta \sin^2(\Theta_{1k} - \Theta_{2k}) \quad \sigma = e, \mu$$

being the expectation value of the number operator on the flavor vacuum.

We thus obtain

$$
\begin{aligned}
S_{ik} &= \sin^2(\Theta_{ik}) \log(\sin^2(\Theta_{ik}) + \cos^2(\Theta_{ik}) \log(\cos^2(\Theta_{ik})) \\
S_{\sigma k} &= \sin^2\theta \sin^2(\Theta_{1k} - \Theta_{2k}) \log(\sin^2\theta \sin^2(\Theta_{1k} - \Theta_{2k})) + \\
&+ (1 - \sin^2\theta \sin^2(\Theta_{1k} - \Theta_{2k}) \log(1 - \sin^2\theta \sin^2(\Theta_{1k} - \Theta_{2k}))
\end{aligned}
$$

which means that, as we expected, the two states have different entropies.

Motivation

- CKM quark mixing, meson mixing, massive neutrino mixing (and oscillations) play a crucial role in phenomenology;
- Theoretical interest: origin of mixing in the Standard Model;
- Bargmann superselection rule[8]: coherent superposition of states with different masses is not allowed in non-relativistic QM;
- Necessity of a QFT treatment: problems in defining Hilbert space for mixed particles[9]; oscillation formulas[10];

[8] V.Bargmann, Ann. Math. (1954); D.M.Greenberger, Phys. Rev. Lett. (2001).

[9] C.W.Kim and A.Pevsner, *Neutrinos in Physics and Astrophysics*, (Harwood, 1993); C.Giunti, J. Phys. G (2007).

[10] M.Beuthe, Phys. Rep. (2003).

Anticommutation, orthonormality and completeness relations

Anticommutation relations:

$$\{\nu_i^{\alpha}(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij} \quad ; \quad \{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta^3(\mathbf{k}-\mathbf{q})\delta_{rs}\delta_{ij}$$

Orthonormality and completeness relations:

$$u^r_{\mathbf{k},i}(t) = e^{-i\omega_{k,i}t}\, u^r_{\mathbf{k},i} \quad ; \quad v^r_{\mathbf{k},i}(t) = e^{i\omega_{k,i}t}\, v^r_{\mathbf{k},i} \quad ; \quad \omega_{k,i} = \sqrt{k^2+m_i^2}$$

$$u^{r\dagger}_{\mathbf{k},i} u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i} v^s_{\mathbf{k},i} = \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},i} v^s_{-\mathbf{k},i} = 0 \quad , \quad \sum_r (u^{r\alpha *}_{\mathbf{k},i} u^{r\beta}_{\mathbf{k},i} + v^{r\alpha *}_{-\mathbf{k},i} v^{r\beta}_{-\mathbf{k},i}) = \delta_{\alpha\beta} \; .$$

Fock space for ν_1, ν_2:

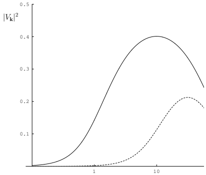
$$\mathcal{H}_{1,2} = \left\{ \alpha^\dagger_{1,2} \;,\; \beta^\dagger_{1,2} \;,\; |0\rangle_{1,2} \right\} .$$

Quantum Field Theory vs. Quantum Mechanics

- Quantum Mechanics:
  - finite $\sharp$ of degrees of freedom
  - unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).*
- Quantum Field Theory:
  - infinite $\sharp$ of degrees of freedom.
  - ∞ many unitarily inequivalent representations of the field algebra $\Leftrightarrow$ many vacua .
  - The mapping between interacting and free fields is "weak", i.e. representation dependent (LSZ formalism)[11]. Example: theories with spontaneous symmetry breaking.

[11] F.Strocchi, *Elements of Quantum Mechanics of Infinite Systems* (World Scientific, 1985).

Condensation density for mixed fermions



- $V_{\mathbf{k}} = 0$ when $m_1 = m_2$ and/or $\theta = 0$.
- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.
- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

In definitive, the flavor fields can be expanded as $((\sigma, i) = (e, 1), (\mu, 2))$:

$$\nu_\sigma(x) \; = \; \sum_{r=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[u^r_{\mathbf{k},i}(t) \alpha^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},i}(t) \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}} \, ,$$

with

$$\alpha^r_{\mathbf{k},\sigma}(t) = G^{-1}_\theta(t) \alpha^r_{\mathbf{k},i} G_\theta(t)$$

$$\beta^r_{\mathbf{k},\sigma}(t) = G^{-1}_\theta(t) \beta^r_{\mathbf{k},i} G_\theta(t).$$

The flavor ladder operators can be also obtained by comparing the field expansions and using:

$$\alpha^r_{\mathbf{k},\sigma}(t) \quad = \quad \int d^3\mathbf{x} \, u^{r\dagger}_{\mathbf{k},i}(x) \, \nu_\sigma(x)$$

$$\beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \quad = \quad \int d^3\mathbf{x} \, v^{r\dagger}_{-\mathbf{k},i}(x) \, \nu_\sigma(x).$$

The “flavor vacuum” $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state[12]:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1-\sin^2\theta\,|V_\mathbf{k}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_\mathbf{k}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_\mathbf{k}||U_\mathbf{k}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_\mathbf{k}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

[12] A. Perelomov, *Generalized Coherent States and Their Applications*, (Springer V., 1986)

Currents and charges for mixed fermions [13]

Lagrangian in the mass basis:

$$\mathcal{L} = \bar{\nu}_m \left(i \not\partial - M_d\right) \nu_m$$

where $\nu_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

[13] M. Blasone, P. Jizba and G. Vitiello, Phys. Lett. B (2001)

$\mathcal{L}$ invariant under global $U(1)$ with conserved charge Q= total charge.
Consider now the $SU(2)$ transformation:

$$\nu'_m = e^{i\alpha_j \tau_j}\, \nu_m \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

The associated currents are:

$$\delta\mathcal{L} \;=\; i\alpha_j\,\bar{\nu}_m\,[\tau_j, M_d]\,\nu_m \;=\; -\alpha_j\,\partial_\mu J^\mu_{m,j}$$

$$J^\mu_{m,j} \;=\; \bar{\nu}_m\,\gamma^\mu\,\tau_j\,\nu_m$$

The charges $Q_{m,j}(t) \equiv \int d^3\mathbf{x}\, J^0_{m,j}(x)$, satisfy the $su(2)$ algebra:

$$[Q_{m,j}(t), Q_{m,k}(t)] \;=\; i\,\epsilon_{jkl}\,Q_{m,l}(t)\,.$$

– The Casimir operator is proportional to the total charge: $C_m = \frac{1}{2}Q$.
$Q_{m,3}$ is conserved $\Rightarrow$ charge conserved separately for ν_1 and ν_2:

$$
\begin{aligned}
Q_1 &= \frac{1}{2}Q + Q_{m,3} = \int d^3\mathbf{x}\, \nu_1^\dagger(x)\, \nu_1(x) \\
Q_2 &= \frac{1}{2}Q - Q_{m,3} = \int d^3\mathbf{x}\, \nu_2^\dagger(x)\, \nu_2(x).
\end{aligned}
$$

These are the flavor charges in the absence of mixing.

The currents in the flavor basis

Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\nu}_f \, (i \not\partial - M) \, \nu_f$$

where $\nu_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

Consider the $SU(2)$ transformation:

$$\nu_f' = e^{i\alpha_j \tau_j} \, \nu_f \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

The charges $Q_{f,j} \equiv \int d^3\mathbf{x}\, J^0_{f,j}$ satisfy the $su(2)$ algebra:

$$[Q_{f,j}(t), Q_{f,k}(t)] = i\,\epsilon_{jkl}\, Q_{f,l}(t).$$

0.2cm
The Casimir operator is proportional to the total charge
$C_f = C_m = \frac{1}{2}Q.$

- $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between ν_e and ν_μ.

Define the flavor charges as:

$$Q_e(t) \equiv \frac{1}{2}Q + Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x)$$

$$Q_\mu(t) \equiv \frac{1}{2}Q - Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

where$Q_e(t) + Q_\mu(t) = Q$.

We have:

$$Q_e(t) = \cos^2\theta\, Q_1 + \sin^2\theta\, Q_2 + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

$$Q_\mu(t) = \sin^2\theta\, Q_1 + \cos^2\theta\, Q_2 - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

In conclusion:

– In presence of mixing, neutrino flavor charges are defined as

$$Q_e(t) \equiv \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x) \quad ; \qquad Q_\mu(t) \equiv \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

– They are not conserved charges $\Rightarrow$ flavor oscillations.

– They are still (approximately) conserved in the vertex $\Rightarrow$ define flavor neutrinos as their eigenstates

- Problem: find the eigenstates of the above charges.

The flavor charge operators are diagonal in the flavor ladder operators:

$$:: Q_{\nu_\sigma}(t) :: \;\equiv\; \int d^3\mathbf{x} \; :: \nu_\sigma^\dagger(x)\, \nu_\sigma(x) ::$$

$$= \; \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^r_{\mathbf{k},\sigma}(t) \,-\, \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^r_{-\mathbf{k},\sigma}(t) \right) , \quad \sigma = e, \mu.$$

Here :: ... :: denotes normal ordering with respect to the flavor vacuum:

$$:: A :: \equiv \; A \,-\, {}_{e,\mu}\langle 0|A|0\rangle_{e,\mu}$$

- Define flavor neutrino states with definite momentum and helicity:

$$|\nu^r_{\mathbf{k},\sigma}\rangle \;\equiv\; \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)\,|0\rangle_{e,\mu}$$

– Such states are eigenstates of the flavor charges (at t=0):

$$:: Q_{\nu_\sigma} :: |\nu^r_{\mathbf{k},\sigma}\rangle \;=\; |\nu^r_{\mathbf{k},\sigma}\rangle$$

– We have, for an electron neutrino state:

$$
\begin{aligned}
\mathcal{Q}_{\mathbf{k},\nu_\sigma}(t) &\equiv \langle \nu^r_{\mathbf{k},e}| \;::\; Q_{\nu_\sigma}(t) \;::\; |\nu^r_{\mathbf{k},e}\rangle \\
&= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2
\end{aligned}
$$

Neutrino oscillation formula (exact result)

[14]:

$$\mathcal{Q}_{\mathbf{k},\nu_e}(t) = 1 - |U_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\, t\right) - |V_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}+\omega_{k,}}{2}\right.$$

$$\mathcal{Q}_{\mathbf{k},\nu_\mu}(t) = |U_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\, t\right) + |V_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}+\omega_{k,}}{2}\right.$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_{\mathbf{k}}|^2 \to 1$ and $|V_{\mathbf{k}}|^2 \to 0$.

[14] M.Blasone, P.Henning and G.Vitiello, Phys. Lett. **B** (1999).